THE SIXTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE sixteenth summer meeting and sixth colloquium of the Society were held at Princeton University during the week September 13–18, 1909. The attendance included the following forty-one members:

Professor G. D. Birkhoff, Professor G. A. Bliss, Mr. R. D. Carmichael, Dr. A. B. Chace, Dr. A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Dr. L. S. Dederick, Professor L. P. Eisenhart, Professor T. C. Esty, Professor H. B. Fine, Dr. G. F. Gundelfinger, Professor Harris Hancock, Rev. A. S. Hawkesworth, Dr. A. M. Hiltebeitel, Dr. J. G. Hun, Dr. Frank Irwin, Professor Edward Kasner, Professor W. R. Longley, Professor J. H. Maclagan-Wedderburn, Mr. H. F. MacNeish, Professor E. H. Moore, Professor Frank Morley, Professor Richard Morris, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Mr. W. J. Risley, Professor D. A. Rothrock, Professor I. J. Schwatt, Dr. Clara E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor Elijah Swift, Professor J. H. Tanner, Mr. C. E. Van Orstrand, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor H. S. White, Professor J. E. Wright, Professor J. W. A. Young.

An account of the colloquium, which opened on Wednesday morning, will appear later in the BULLETIN. The four sessions of the summer meeting proper occupied the first two days of the week. At the opening session Professor Fine occupied the chair, which was taken at the later sessions by Professor Morley and Vice-Presidents Kasner and Van Vleck. The Council announced the election of the following persons to membership in the Society: Dr. L. S. Dederick, Princeton University; Dr. G. E. Wahlin, University of Illinois; Mr. E. E. Whitford, College of the City of New York. Eleven applications for membership were received.

The sum of five thousand francs was appropriated from the treasury of the Society in support of the publication of the works of Euler and in subscription for a copy of these works for the Society's library. A committee was appointed to prepare and report at the October meeting suitable resolutions on the death of Ex-President Simon Newcomb. The committee

on nominations, consisting of Professors E. B. Van Vleck, Brown, and Kasner, appointed to nominate officers and other members of the Council will also report at the October meeting.

At the closing session resolutions thanking the University for its hospitality and the committee on arrangements for their services were adopted. On Tuesday the members were conducted through the buildings and grounds of the University, and Tuesday evening was marked by a reception at the house of Professor Fine.

The following papers were read at this meeting:

(1) Professor L. P. EISENHART: "Congruences of the elliptic type."

(2) Mr. DUNHAM JACKSON: "Resolution into involutory substitutions of the transformation of a bilinear form into itself."

(3) Dr. F. W. REED: "On singular points in the approximate development of the perturbative function."

(4) Professor VIRGIL SNYDER: "Surfaces invariant under infinite discontinuous birational groups defined by line congruences."

(5) Mr. JOSEPH LIPKE: "Natural families of curves in a general curved space," preliminary communication.

(6) Rev. A. S. HAWKESWORTH: "A new theorem in conics."

(7) Mrs. ANNA J. PELL: "Applications of biorthogonal systems to integral equations."

(8) Mr. G. C. EVANS: "The integral equation of the second kind, of Volterra, with singular kernel," preliminary communication.

(9) Professor EDWARD KASNER: "Triply orthogonal systems of surfaces."

(10) Professor EDWARD KASNER: "Natural families and Thomson's theorem."

(11) Professor G. A. MILLER: "The groups which may be generated by two operators s_1, s_2 satisfying the equation $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$, α and β being relatively prime."

(12) Dr. F. R. SHARPE: "Integral equations with variable limits, with an application to the problem of age distribution."

(13) Mr. R. D. CARMICHAEL: "Note on a new number theory function."

(14) Professor T. E. MCKINNEY: "On a criterion for λ-developments in the theory of equivalence."

(15) Dr. G. G. CHAMBERS: "Groups of isomorphisms of the abstract groups of order p^2q."

(16) Professor W. R. LONGLEY: "Note on some periodic orbits with more than one axis of symmetry."

(17) Professor W. H. BUSSEY: "Tables of Galois fields of order less than 1,000."

(18) Professor W. B. FORD: "On the relation between the sum formulas of Hölder and Cesàro."

(19) Professor OSWALD VEBLEN: "Products of pairs of involutoric projectivities."

(20) Dr. G. F. GUNDELFINGER: "On the geometry of line elements in the plane with reference to osculating vertical parabolas and circles."

(21) Professor P. F. SMITH: "Theorems in the geometry of surface elements in space."

(22) Professor R. G. D. RICHARDSON and Mr. W. A. HURWITZ: "Note on determinants whose terms are certain integrals."

(23) Professor R. G. D. RICHARDSON: "The Jacobi criterion in the calculus of variations and the oscillation of solutions of linear differential equations of the second order."

(24) Professor I. J. SCHWATT: "Methods for the summation of infinite series."

(25) Professor A. B. COBLE: "Cubic space curves that meet the Hessian of a cubic surface in six pairs of corresponding points."

(26) Professor G. D. BIRKHOFF: "On the theory of stability."

(27) Mr. H. W. REDDICK: "Geometric properties of a system of tautochrones."

(28) Dr. W. B. CARVER: "The poles of finite groups of fractional linear substitutions in the complex plane."

(29) Dr. L. S. DEDERICK: "The solution of the equation in two real variables at a point where both the partial derivatives vanish."

(30) Dr. H. T. BURGESS: "On point-circle correlations in the plane."

(31) Professor H. B. NEWSON: "A general theory of linear groups."

(32) Mr. A. R. SCHWEITZER: "A formal extension of Bolzano's series."

Mr. Jackson's paper was communicated to the Society by Professor Bôcher, Dr. Reed's by Professor White, and Mr. Evans's by Professor Osgood. Dr. Burgess was introduced by Professor Osgood. The papers of Dr. Sharpe and Dr. Carver

were presented by Professor Snyder. The papers of Mr. Jackson, Dr. Reed, Mr. Lipke, Mrs. Pell, Mr. Evans, Professor Kasner's second paper, and the papers of Professor Miller, Professor McKinney, Dr. Chambers, Professor Bussey, Professor Ford, Professor Richardson and Mr. Hurwitz, Professor Richardson, Professor Coble, Professor Newson, and Mr. Schweitzer were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Kummer in his study of rectilinear congruences made use of two quadratic differential forms, namely $Edu^2 + 2Fdudv + Gdv^2$, $edu^2 + (f + f')dudv + gdv^2$, of which the first is the square of the linear element of the spherical representation of the congruence. When the Jacobian of these two forms, written $Adu^2 + 2Bdudv + Cdv^2$, is equated to zero, the resulting differential equation defines the developable ruled surfaces of the congruence. This equation admits of real solutions in the case of normal congruences, congruences of Guichard, cyclic congruences, and congruences of tangents to a real surface. But there is a large variety of congruences for which the solutions of this equation are imaginary, and yet little or no study has been made of them. It is with these congruences of the elliptic type that Professor Eisenhart's paper deals. By means of a theorem of Cifarelli it is found that through each line of a congruence there pass two real ruled surfaces such that when they are parametric, the relations $A/E = C/G$, $B = 0$ obtain; they are called the characteristic ruled surfaces of the congruence. These ruled surfaces are found to possess properties analogous to certain well-known properties of the developable surfaces of a congruence of the hyperbolic type.

For a congruence of the elliptic type the abscissas of the focal points, measured from the middle point, are conjugate pure imaginary quantities. If the latter be denoted by ρ and $-\rho$, the locus of the points at the distances $i\rho$ and $-i\rho$ constitute a pair of real surfaces which are called the pseudo-focal surfaces. When these surfaces are associate to one another, the congruence is of the kind discussed by Lilienthal. A further study of these congruences is made in terms of parameters so chosen that $A = C$, $B = 0$, and these results are considered in connection with the congruences formed by the joins of corresponding points on associate surfaces. The paper closes with a discussion

of the congruences whose developables cut the middle surface in the unicursal curves on the latter.

2. In Mr. Jackson's paper it is first pointed out that by combining a theorem recently proved by E. B. Wilson with an old result of Frobenius one can at once infer that a necessary and sufficient condition that a linear transformation be resolvable into the product of two involutory transformations is that there exist a non-singular bilinear form which this transformation carries over into itself. A new proof of the theorem of Wilson just referred to is then given.

3. Poincaré has shown how the higher terms in the development of the perturbative function may be derived approximately without computing the intervening terms. Applications of the theory to particular cases, principally in coplanar orbits, have been made by Coculesco, Féraud, and Hamy. Dr. Reed's paper considers the general case of elliptic orbits with any mutual inclination. The method is in part dependent upon numerical assumptions, but by use of a criterion for the admissibility of those singular points upon which the approximations depend the application can be made to orbits of any elements whatever. The solution of an ideal Jupiter-asteroid problem of general type is given in detail.

4. If the equation of a quadric contains a rational parameter to some degree higher than the first, and the two systems of generators are rationally separable, each system Σ, T will define a line congruence having the envelope of the family for focal surface ϕ. Each quadric will touch the focal surface in the points of an elliptic space quartic C_4. If a generator of Σ touches ϕ in P_1, P_2, the operation of interchanging P_1, P_2 is involutional. Through P_2 passes a tangent to ϕ belonging to T, touching ϕ again in Q_2. The operation ΣT is now of infinite order.

For special relations among the coefficients, one, two, or three linear factors can be removed from ϕ. Professor Snyder established the following theorem:

Surfaces of order four or any number greater than five can be constructed which are invariant under an infinite discontinuous group of birational transformations.

All the congruences of order 2 except those of class 7 are shown to belong to this category. The method is applied to

two other congruences and the fundamental points of the transformation are determined.

5. In a recent article in the *Transactions*,* a complete geometric characterization of the families of curves defined as the extremals connected with variation problems of the form $\int F ds = minimum$ (where F is any point function and ds is the element of arc in the space considered) is given for euclidean spaces of two and three dimensions; this may be easily extended to all spaces of constant curvature. In the present preliminary paper, Mr. Lipke gives a complete geometric characterization of natural families of curves in a curved space of two dimensions (general surface). The ∞^2 curves of any natural family in such a space have the following properties: (A) The locus of the centers of geodesic curvature of the ∞^1 curves through a point is a straight line; (B) among the ∞^1 geodesic circles (circles of constant geodesic curvature) which osculate the ∞^1 curves through a point, there exist two hyperosculating circles which are orthogonal. The natural families are merely a special case of a much broader class of curves characterized by property (A). The latter class also includes families of isogonal trajectories, which may be associated with and are gotten from the natural families by a curvature transformation.

6. Mr. Hawkesworth's paper is in abstract as follows: If two or more triangles to which any conic is in common escribed have each a vertex upon the same axis, and if in each case a circle be described passing through the triangle's other two vertices and with its center upon the axis, then the resultant system of circles will have the conic's other axis for their common radical axis passing in common through the two foci, or focoids, as the case may be, lying upon it, while their "limiting points" upon the first and given axis and line of their centers will be, in turn, the conic's focoids or foci. With other corollaries, the very important one follows that if any triangle cut by any transversal be given, then there is an easily found unique conic that has the transversal for an axis and is circumscribed by the triangle.

7. By considering a given complete biorthogonal system in

* "Natural families of trajectories: conservative fields of force." By Edward Kasner, *Trans. Amer. Math. Society*, April, 1909.

connection with some of Hilbert's methods Mrs. Pell obtains the following results: Either the no -homogeneous integral equation

$$\phi(s) = f(s) + \int_a^b K(s,t)\psi(t)dt,$$

where $\phi(s)$ and $\psi(s)$ are to be determined as associated functions with respect to the given biorthogonal system, has a solution $[\phi(s), \psi(s)]$; or the homogeneous integral equation

$$\phi(s) = \int_a^b K(s,t)\psi(t)dt$$

has. If the unsymmetrical kernel $L(s,t)$ is the associated function of a symmetrical kernel there always exist characteristic values λ_i and corresponding characteristic solutions

$$\phi_i(s) = \lambda_i \int_a^b L(s,t)\phi_i(t)dt, \qquad \psi_i(s) = \lambda_i \int_a^b L(t,s)\psi_i(t)dt.$$

The solutions $[\phi_i(s), \psi_i(s)]$ form a biorthogonal system equivalent to the given one. Any function $f(s)$ expressible in the form

$$f(s) = \int_a^b L(s,t)f_1(t)dt$$

can be developed in a uniformly convergent series of the form

$$f(s) = \sum_{i=1}^{\infty} \phi_i(s) \int_a^b \psi_i(s)f(s)ds.$$

9. Professor Kasner obtains a set of general relations involving the curvatures of the three surfaces and the curvatures of of their curves of intersection. The more general theory of triply orthogonal congruences of curves is also studied. Corollaries are given for plane curves.

10. In Professor Kasner's second paper it is shown that the orthogonality properties involved in Thomson's theorem are sufficient to characterize natural families of trajectories. Stronger converse results are also obtained.

11. Professor Miller's paper appears in full in the present number of the BULLETIN.

12. The integral equation of the second kind with variable limits which have a constant difference has been solved by P. Hertz and C. Herglotz (*Mathematische Annalen*, volume 65). In Dr. Sharpe's paper a more direct and shorter solution is obtained by considering the limit of a finite difference equation with constant coefficients. The complete solution is found for the case in which a certain transcendental equation has equal roots. An application is made to the problem of the age distribution of population. It is shown that the final distribution is ultimately independent of the initial distribution.

13. By means of Euler's ϕ-function Mr. Carmichael defines $\lambda(n)$ as follows: $\lambda(p^a) = \phi(p^a)$ when p is an odd prime; $\lambda(2^a) = \phi(2^a)$ when $a = 1$ or 2; $\lambda(2^a) = \frac{1}{2}\phi(2^a)$ when $a > 2$; $\lambda(2^a p_1^{a_1} \cdots p_i^{a_i}) =$ the lowest common multiple of $\lambda(2^a)$, $\lambda(p_1^{a_1}), \cdots, \lambda(p_i^{a_i})$, $p_1, \cdots, p_i$ being different odd primes. If $\lambda(n)$ is the exponent to which a belongs modulo n, then a is called a primitive λ-root of $x^{\lambda(n)} = 1 \pmod{n}$. The following are some of the theorems obtained:

For any given n the congruence $x^{\lambda(n)} \equiv 1 \pmod{n}$ is satisfied by every x prime to n. In every congruence $x^{\lambda(n)} \equiv 1 \pmod{n}$ a solution g exists which is a primitive λ-root, and for any such g there are $\phi\{\lambda(n)\}$ primitive λ-roots congruent to powers of g. If $\lambda(n) > 2$ the product of primitive λ-roots congruent to powers of g is congruent to 1 (mod n). If x_1 is the largest value of x satisfying the equation $\lambda(x) = a$, any other solution x_2 is a factor of x_1. Let a be that divisor of α for which $\lambda(x) = a$ has a greatest solution x_1 greater than such a solution when for a any other divisor of α is taken. Then x_1 is the largest divisor of $z^a - 1$ for every z prime to the divisor. Finally it is shown that there are values of composite n for which the relation $e^{n-1} \equiv 1 \pmod{n}$ is true when e is any number prime to n, and a method is given for finding such values of n.

14. In a paper "Concerning a certain type of continued fractions depending on a variable parameter" in the *American Journal of Mathematics*, volume 29, number 3, Professor McKinney showed that when X_0 is properly equivalent to a "critical value" x_0, then $\lambda(X_0)_{\lambda'}$ may be ultimately like $\lambda(x_0)_{\lambda'}$ or ultimately like the "associate" of $\lambda(x_0)_{\lambda'}$. In that paper no criterion was established for deciding between these two cases. It is the object of the present paper to set up a criterion for an important class of critical values.

15. The paper by Dr. Chambers takes up the non-abelian abstract groups of order p^2q as determined by Hölder, and finds for each its corresponding group of isomorphisms and the group of cogredient isomorphisms. Independent generators for these groups of isomorphisms are obtained in terms of the generators of the original groups, and sufficient relations among these generators are determined to define the groups of isomorphisms.

16. The discussion of the periodic orbits of a particle which is subject to the newtonian attraction of finite bodies whose motion is supposed to be known, begins naturally with Jacobi's restricted problem of three bodies. In this case all the periodic orbits of the particle in the plane of motion of the finite bodies have one axis of symmetry, namely, the line joining the finite bodies. The symmetrical properties are important, both in the numerical processes of Darwin * and the analysis employed by Moulton.† In generalizing the problem by introducing a greater number of finite bodies, the next step is to consider three spheres moving in circles according to the equilateral triangle solution of Lagrange. This problem has been treated in detail.‡ The periodic orbits of the particle have no axis of symmetry, or they have one such axis, depending upon the distribution of mass. Professor Longley's note considers certain cases in which the particle is subject to the attraction of more than three finite bodies revolving in circles about their center of mass. Some of the periodic orbits of the particle about one of the finite bodies have more than one axis of symmetry, and, as a consequence, it is possible to make some simplifications in the analysis used for the preceding cases.

17. In the BULLETIN, volume 12 (1905), pages 21–38, Professor Bussey published a paper entitled "Galois field tables for $p^n \leqq 169$." The present paper is an extension of that work and contains tables of all Galois fields of order p^n such that $n > 1$ and $169 < p^n < 1{,}000$. The limit 1,000 is that set by Jacobi in his Canon Arithmeticus, which contains tables of indices for all primes less than 1,000. These tables of indices are tables of all Galois fields of prime order. When this paper is published, there will be in print tables of all Galois fields of order less than 1,000.

* *Acta Mathematica*, vol. 21 (1897), p. 99.
† *Transactions Amer. Math. Society*, vol. 7 (1906), p. 537.
‡ *Ibid.*, vol. 8 (1907), p. 159.

18. Professor Ford's paper is devoted to a proof of the folfowing theorem: "If a given series (convergent or divergent) is summable and of indeterminacy r in Hölder's sense, it is summable and of indeterminacy r in Cesàro's sense, and conversely." The direct part of this theorem has already been established by Knopp in his dissertation "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze" (Berlin, 1907), while the converse has been proved for the special cases in which $r = 1$, 2 by Bromwich (*Mathematische Annalen*, volume 65 (1908), pages 363–365). The above theorem supplements these investigations by establishing both the direct and converse theorems for all values of r. It follows that the sum formulas of Hölder and Cesàro are coextensive in their applicability to given series and determine one and the same sum.

The paper has been offered to the *American Journal of Mathematics*.

19. Professor Veblen's paper has to do with a class of theorems of which Pascal's theorem on conic sections is one and of which another is the following: If a simple hexagon is inscribed in a surface of the second degree, each pair of opposite sides is met by a pair of lines conjugate to both sides, and the latter three pairs of lines are met by one pair of lines.

20. The investigations made by Dr. Gundelfinger in this paper arise from an interpretation in the plane of the geometry within a linear line complex by means of a transformation due to Sophus Lie.* This transformation takes a point of space into a line element of the plane; curves of the complex into unions; complex lines, in particular, into vertical parabolas; — hence the tangents to a complex curve transform into the vertical parabolas which osculate the corresponding union.

From a classification of curves and of surfaces with respect to the group G_{10} of projective transformations in space which leave the linear line complex invariant we obtain a classification of "line element loci" of ∞^1 and of ∞^2 elements with respect to the Γ_{10} of contact transformations in the plane — that is we classify ordinary differential equations of the first order with respect to the contact relations of the ∞^2 vertical parabolas which osculate their integral curves.

The fact that the linear line complex is invariant also under the correlation which it defines when regarded as a null system

gives rise to a theory of "reciprocal line element loci" of much interest. Furthermore, the various types of congruences of complex lines, characterized by the nature of their focal surface the two sheets of which are reciprocal configurations, give rise to all possible arrangements of ∞^2 vertical parabolas in the plane which are distinct in regard to the singular solution of the corresponding differential equation of the second order.

By using a second transformation of Lie's † which differs from the above in that complex lines go over into circles instead of vertical parabolas, a second interpretation is effected giving rise to many interesting theorems on the osculating circle.

21. In recent investigations of Scheffers and Lilienthal some interesting theorems are demonstrated on the osculating circles of the integral curves of an ordinary differential equation of the first order. Professor Smith's paper recasts these results in such a way that generalization to partial differential equations is made possible. It is found that the rôle played by the osculating circles is assumed in space by osculating vertical parabolic bands, that is, bands on the surface of a parabolic cylinder between any two parallel vertical planes indefinitely near. It is shown that the osculating vertical parabolic bands of the characteristic bands of an arbitrary partial differential equation of the first order belong to a second partial differential equation — the reciprocal of the former. Exceptional cases are easily characterized. The results follow readily from the projective geometry within a null-system in five-dimensional space, a subject which has recently received the attention of Professor Eiesland.

22. The paper by Professor Richardson and Mr. Hurwitz appeared in full in the October number of the BULLETIN.

23. Hilbert has recently discovered * the connection between the calculus of variations and the theories of differential and integral equations and has emphasized the fact that the former theory is the wider in its scope. The self-adjoint differential equation of the second order $(pu')' + qu + \lambda ku = 0$ containing the parameter λ may be regarded as the Lagrange equation of a calculus of variation problem with a certain quadratic sub-condition or with this quadratic and certain linear sub-conditions. The questions concerning the necessary and sufficient

* Geometrie der Berührungstransformationen, p. 238.
† Ibid., p. 245.

conditions for the existence of a minimum, in particular the significance of the Jacobi criterion are discussed in this paper by Professor Richardson. The chief theorem may be stated as follows: The Jacobi criterion demands that the solution $u(x) = U_1(x)$ of the problem

$$\int_0^1 (pu'^2 - qu^2)dx = min, [p(x) > 0,\ q(x) \leqq 0,\ u(0) = u(1) = 0],$$

with the quadratic sub-condition

$$\int_0^1 k(x)u^2dx = 1,$$

shall not vanish in the interval (0, 1), that the solution $u(x) = U_2(x)$ of the same problem with the quadratic and one linear sub-condition

$$\int_0^1 kU_1(x)u(x)dx = 0$$

shall vanish once in the interval, and in general that the solution $u(x) = U_{n+1}(x)$ of the problem with the quadratic and n linear sub-conditions

$$\int_0^1 k(x)U_1(x)u(x)dx = 0,\ \cdots,\ \int_0^1 k(x)U_n(x)u(x)dx = 0$$

shall vanish exactly n times in the interval.

This paper is the first of a series to appear in the *Mathematische Annalen.*

24. Professor Schwatt developed certain methods for the summation of special form of infinite series.

25. Professor Coble's paper deals with two kinds of cubic space curves associated with a general cubic surface.

The curve of the one kind has six skew lines of a double six of the surface as axes. It contains ∞^2 sets of five osculating planes whose ten meets are on the surface. There are 72 systems of these curves, each system containing ∞^2 curves.

The curve of the other kind meets the Hessian of the surface in six pairs of corresponding points. It contains ∞^2 sets of five

*"Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen"; *Göttinger Nachrichten,* 1 und 2 Mittheilungen 1904, 4 und 5 Mittheilungen 1906.

points any three of which are an apolar triad of the surface. There are 16 systems of these curves, each system containing the ∞^2 curves on a set of five points found among the poles of the planes of the Sylvester pentahedron.

26. A solution $(x_1, x_2, \cdots, x_n)$ of the system of ordinary differential equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \cdots, x_n) \quad (i = 1, 2, \cdots, n)$$

is defined to be stable if $x_1, x_2, \cdots, x_n$ remain finite for $t > t_0$. This paper by Professor Birkhoff is a study of such solutions, especially with reference to their periodic or quasi-periodic character.

27. The totality of tautochrones in a plane, due to a force whose rectangular components are functions only of the coordinates of the point at which the force acts, forms a system of ∞^3 curves. In the July number of the BULLETIN Professor Kasner gave the intrinsic equation of such a system. In the present paper Mr. Reddick deduces the cartesian equation of the system, a differential equation of the third order, and investigates its geometric properties. The necessary and sufficient conditions that any plane system of ∞^3 curves shall be a family of tautochrones are obtained.

28. In his Lectures on the icosahedron, Klein shows that any finite group of fractional linear substitutions on a complex variable can be reduced to a canonical form such that each substitution will correspond (by the ordinary stereographic projection) to a rotation of the sphere which has the unit circle of the complex plane as a great circle. The poles of each substitution are what we may call "skew-inverse" points with respect to the unit circle.

The following questions, which are in fact equivalent, naturally arise:

(1) If a finite group of fractional linear substitutions is *not* in this canonical form, do its operations correspond in the same way to rotations of a sphere having some other circle of the complex plane as a great circle?

(2) If upon all pairs of skew-inverse points with respect to a circle in the complex plane, one operates with an arbitrary fractional linear substitution, will they go into skew-inverse points with respect to some circle; and if so, what circle?

In the present paper Dr. Carver answers the second question; and hence also the first.

29. In an earlier paper,* Dr. Dederick gave sufficient conditions for the existence of a solution of the equation $F(x, y) = 0$, in the form $y = f(x)$ in the neighborhood of a point where $\partial F/\partial x = \partial F/\partial y = 0$, and for the existence and continuity of dy/dx in the case where the equation for formally determining dy/dx has a simple root. The present paper extends these results to the higher derivatives of y with regard to x, and to the case of a multiple root of the equation for dy/dx. The results may be summed up in the statement: The ordinary process for determining the character of the branches of the plane curve $F(x, y) = 0$ at a singularity, and the method of differentiating $F(x, y)$ totally with regard to x or with regard to a parameter t, in order to obtain the successive derivatives of y with regard to x or of x and y with regard to t, are justified if the partial derivatives of $F(x, y)$ which are used are continuous at and near the point in question.

30. The problem of the determination and discussion of point-circle correlations in the plane from the standpoint of the geometry of contact transformations is treated in Dr. Burgess's paper. The method used involves those questions in the geometry on a quadric which arise in point-plane correlations in space. Algebraic difficulties make it necessary to limit the scope of the paper to a complete solution of the involutory cases — a solution effected by elementary divisors. The Cayley numbers for a curve on a quadric are interpreted for its stereographic projection, and formulas given by which the Cayley numbers for the transformed curve may be found.

The æquatio-directrix of the transformation suggests two problems: (1) the study of the properties of a single transformation, and (2) the study of a system of transformations in one parameter. In certain cases, this system turns out to be a group.

31. A general theory of linear groups which applies equally to continuous and discontinuous groups in any number of variables was announced in Professor Newson's paper. If no restrictions are laid upon the variation of the elements of the matrix M of a linear transformation T in n variables, the continuous group G_{n^2-1} of ∞^{n^2-1} collineations in space of $n-1$

* Presented at the April meeting of the Society.

dimensions is obtained. A subgroup of G_{n^2-1} is obtained when the elements of M satisfy certain conditions, as *e. g.*, the well-known conditions defining the orthogonal group. Professor Newson's fundamental theorem lays down the necessary and sufficient conditions which its elements of M must satisfy in order to have a subgroup of G_{n^2-1}. He defines a complete family of automorphic forms ϕ_i which are homogeneous and symmetric functions in from 1 to n sets of n variables each. His theorem is: A necessary and sufficient condition for the existence of a subgroup of G_{n^2-1} is that the elements of M satisfy a set of equations $\phi_i = l_i$ consisting of a complete family of automorphic forms in the elements of the rows or columns of M, each equated to the corresponding coefficient of the family.

Families of lower degrees define continuous subgroups of G_{n^2-1}; after a certain degree is reached the subgroups become discontinuous; above a certain other degree the conditions are satisfied only by the identical transformation. The paper will be published in the *Kansas University Science Bulletin.*

32. Mr. Schweitzer contrasted the formal properties of Bolzano's linear series with his exposition of the series of Vailati (the system ${}^1R_1$) and showed how to extend Bolzano's series to n dimensions ($n = 1, 2, 3, \cdots$) by considering simple modifications of the axioms of dimensionality and extension in his system ${}^nR_n$. Application of the author's n-dimensional open and closed chains is made.

F. N. COLE,
Secretary.

THE GROUPS WHICH MAY BE GENERATED BY TWO OPERATORS s_1, s_2 SATISFYING THE EQUATION $(s_1s_2)^\alpha = (s_2s_1)^\beta$, α AND β BEING RELATIVELY PRIME.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 13, 1909.)

SINCE s_1s_2 and s_2s_1 are of the same order and α, β are relatively prime, it results that this common order is prime to both α and β. Hence s_1s_2 and s_2s_1 are generated by either $(s_1s_2)^\alpha$ or $(s_2s_1)^\beta$, and the cyclic group generated by s_1s_2 coincides with the one generated by s_2s_1. A direct consequence of this is that the group generated by s_1s_2 is invariant under the entire group G

generated by s_1, s_2. It is also evident that G is generated by $s_1 s_2$, s_1, and that every group which involves an invariant cyclic subgroup and is generated by this subgroup and an additional operator may also be generated by two operators satisfying the conditions imposed upon s_1, s_2 in the heading of this paper. These facts establish the theorem:

The totality of the groups which may be generated by two operators satisfying the condition $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$, α *and* β *being relatively prime, coincides with the totality of those generated by two operators which are such that one of them is transformed into a power of itself by the other.*

The groups which may be generated by two operators t_1, t_2 having a common square have been called the generalized dihedral rotation groups.* That these groups are included in the category satisfying the conditions imposed in the heading of the present paper results directly from the facts that

$$t_1 t_2^{-1} = (t_2^{-1} t_1)^{-1}, \quad \{t_1, t_2\} \equiv \{t_1, t_2^{-1}\}.$$

Hence this category of groups may be regarded as composed of those groups which result from a second generalization of the dihedral rotation groups; the first generalization corresponds to the special values $\alpha = 1$, $\beta = -1$. As $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$ expresses only one condition between s_1, s_2 this equation must be satisfied by the generators of an infinite number of distinct groups of finite order for every pair of values of α, β.†

One of the most interesting features of the relation $(s_1 s_2)^\alpha = (s_2 s_1)^\beta$ is that it may be used as one of the two conditions which are satisfied by the two generators of only a finite number of groups. This fact may be established as follows: If $s_1^a = s_2^b$, it is evident that s_1^a is invariant under G and hence $\{s_1^a, s_1 s_2\}$ is an abelian group. When a and b are relatively prime this abelian group coincides with G; for otherwise the quotient group of G with respect to this abelian group would have an order prime to b, and s_2 would correspond to an operator differing from the identity in this quotient group. This is impossible since s_2^b would have to correspond to identity. Having proved that G is abelian whenever α, β and a, b are two pairs of relatively prime numbers, it is not difficult to prove that there is only a finite number of distinct operators which can satisfy both of these conditions for given values of

* *Archiv der Mathematik und Physik*, vol. 9 (1905), p. 6.
† *Amer. Jour. of Mathematics*, vol. 31 (1909), p. 167.

the relatively prime pairs α, β and a, b. We shall prove this fact in the following paragraph.

Since G is abelian $(s_1s_2)^\alpha = (s_2s_1)^\beta = (s_1s_2)^\beta$, and hence $(s_1s_2)^{\alpha-\beta} = s_1^{\alpha-\beta}\, s_2^{\alpha-\beta} = 1$, or $s_1^{\alpha-\beta} = s_2^{\beta-\alpha}$. Combining this equation with $s_1^a = s_2^b$, there results $s_1^{a(\alpha-\beta)} = s_2^{a(\beta-\alpha)} = s_2^{b(\alpha-\beta)}$, and hence

$$s_2^{(a+b)(\beta-\alpha)} = 1 = s_1^{(a+b)(\alpha-\beta)}.$$

As the orders of s_1, s_2 are limited and these operators must be commutative, this proves that *only a finite number of groups can be generated by two operators which satisfy both of the equations*

$$(s_1s_2)^\alpha = (s_2s_1)^\beta \quad \text{and} \quad s_1^a = s_2^b,$$

where α, β and a, b represent two pairs of relatively prime numbers. For instance, when these numbers are 4, 5 and 2, 3 G is the group of order 5. That is, if s_1, s_2 satisfy both of the equations

$$(s_1s_2)^4 = (s_2s_1)^5, \quad s_1^2 = s_2^3,$$

they must generate the group of order 5. This result establishes close contact between the present note and the paper "On groups which may be defined by two operators satisfying two conditions," *American Journal of Mathematics*, volume 31 (1909), page 167.

A NOTE ON IMAGINARY INTERSECTIONS.

BY PROFESSOR ELLERY W. DAVIS.

In the plane let there be a conic C and a line L. Set up a system of coordinates such that L is the line infinity, its pole O with regard to C is the origin, the axes OX and OY are conjugate with regard to C, while X and Y are their intersections with L. Furthermore let $x = \pm 1$, $y = \pm 1$ be tangents to C through Y and X respectively. Then $x =$ a constant passes through Y, while $y =$ a constant passes through x. All these lines are to be determined by the fact that any four convergents form a harmonic set when the constants in the right member are a harmonic set of numbers. In brief, C, L, and the coordinates are projectively transformed from a circle $x^2 + y^2 = 1$, the line infinity, and a rectangular system whose origin is the center of the circle. The equation of any line in the transformed coordinates is precisely the same as that of which it is the projection in the rectangular coordinates.

In the transformed coordinates, let us represent the point with complex coordinates $(x' + ix'', y' + iy'')$ by a vector from (x', y') to $(x' + x'', y' + y'')$. If we change to another pair of conjugate axes OX_1, OY_1 the coordinates are changed to $x_1' + ix_1''$, $y_1' + iy_1''$, but the vector is left unchanged since its end points are.

Consider, however, the intersections of C with L. These are given by $x' = \infty$, $x'' = 0$, $y' = 0$, $y'' = \pm ix'$. That is, the points x', $\pm y''$ are the intersections of $y = \pm x$ with L.

But the lines $y = \pm x$ change when we change the axes as above. Thus the vectors taken to represent the intersections of C with L also change.

If with Cauchy we represent $(x' + ix'', y' + iy'')$ by a vector joining the real points on the circular rays through it, we meet the same difficulty. For these real points are $(x' + y'', y' - x'')$ and $(x' - y'', y' + x'')$; that is, in the case under consideration, they are $(\infty, 0)$ and $(0, 0)$ so that the direction changes as OX does.

Von Staudt's representation is, however, unimpaired; since his imaginary intersection is the involution of all the point pairs cut out on L by the involution of all the line pairs $y = \pm x$. The vector representation should then be regarded as a symbol, adapted to the particular coordinates used, for the more complete Von Staudt representation.

UNIVERSITY OF NEBRASKA,
July 16, 1909.

MAUROLYCUS, THE FIRST DISCOVERER OF THE PRINCIPLE OF MATHEMATICAL INDUCTION.

BY DR. G. VACCA.

Introductory Note. — Soon after the publication of my review of Voss's address (BULLETIN, volume 15, page 405), wherein I considered at some length the history of mathematical induction, I received a note from Professor Moritz Cantor of Heidelberg, in which he called my attention to Dr. Vacca's research on this same historical topic. As Dr. Vacca's research was not accessible to me, I wrote to him for information and received, in reply, the following notes, which will doubtless be of general interest to American readers. — FLORIAN CAJORI.

Many years ago I published in the *Formulaire de Mathé-*

matique of Professor Peano an account of the first discovery of mathematical induction, as due to the Italian Maurolycus. But this paper seems to have had only a small diffusion. I think it useful, therefore, to give, with some more details, a short account of this important discovery.

I.

Franciscus Maurolycus was born in Messina in the year 1494, and died in the same city in the year 1575. His works on the Greek mathematical authors, Euclid, Archimedes, Apollonius, Theodosius, and many others, have been of the greatest importance for the transmission of Greek science to Europe. But the most original of his works is the treatise on arithmetic "Arithmeticorum libri duo" written in the year 1557 and printed in Venice in the year 1575 in the collection "D. FRANCISCI MAUROLYCI Opuscula mathematica."

In the Prolegomena to this work he points out that neither in Euclid nor in any other Greek or Latin writer (among them he enumerates Iamblichus, Nicomachus, Boetius) is there, to his knowledge, a treatment of the polygonal and polyhedral numbers, and he reproaches Jordanus for having been content with a useless repetition of what was written by Euclid.

"Nos igitur [he says] conabimur ea, quae super hisce numerariis formis nobis occurrunt, exponere: multa interim faciliori via demonstrantes, et ab aliis authoribus aut neglecta, aut non animadversa supplentes."

This new and easy way is nothing else than the principle of mathematical induction. This principle is used at the beginning of the work only in the demonstration of very simple propositions, but in the course of the treatise is applied to the more complicated theorems in a systematic way.

For instance, he demonstrates at first that: "omnis quadratus cum impari sequente coniunctus, constituit quadratum sequentem (Prop. 13)." [In modern symbols: if a is a number, then $a^2 + (2a + 1) = (a + 1)^2$].

Using this result, he now demonstrates that:

"Ex aggregatione imparium numerorum ab unitate per ordinem successive sumptorum construuntur quadrati numeri continuati ab unitate, ipsisque imparibus collaterales (Prop. 15)" [in modern symbols:

$$1 + 3 + 5 + 7 + \cdots + (2a + 1) = (a + 1)^2].$$

"Nam per P 13, unitas imprimis cum impari sequente, facit quadratum sequentem, scilicet 4. Et ipse 4 quadratus secundus, cum impari tertio, scilicet 5, facit quadratum tertium, scilicet 9. Itemque 9 quadratus tertius, cum impari quarto, scilicet 7, facit quadratum quartum, scilicet 16. Et sic deinceps in infinitum, semper P 13 repetitam propositum demonstratur."

The form adopted by Maurolycus with the object of persuading the reader of the truth of his demonstrations is generally the following. He applies the reasoning to the first numbers, very often to the first five, and then he concludes with some one of these phrases:

"et eodem syllogismo pro quovis alio assignato loco, utemur ad roborationem propositi (pag. 30, Prop. 65)."

"et argumentatio a quinto loco ad alia quaevis loca transferetur ad conclusionem propositi (pag. 31, Prop. 66).

"et a quinto loco transfertur syllogismus ad quemvis alium, ut propositio conclusit (pag. 33, Prop. 67)."

There are in this work many other points of interest for the history of mathematical knowledge, but I hope to be able to write something on it at another occasion.

II.

But now we have a question before us. Was Pascal unaware of the book of Maurolycus?

In his Traité du triangle arithmétique, printed perhaps in the year 1657, he never mentions Maurolycus, notwithstanding that, in my opinion, this treatise is only an application of the method discovered by Maurolycus. But Pascal, shortly after, being engaged in the polemic concerning the cycloid, in the well-known letter "Lettre de Dettonville à Carcavi" had to demonstrate a proposition concerning the triangular and pyramidal numbers. He says then: *

"Cela est aisé par Maurolic."

It is strange to point out that not even the name of Maurolycus has been included in the Table analytique of the old edition of the works of Pascal, and more strange that the editors of the new edition of the "Oeuvres" of Pascal † in a very incomplete

* B. Pascal, Oeuvres complètes, tome troisième, Paris, Hachette, 1889, page 377.

† Blaise Pascal, Oeuvres par Léon Brunschwigg et Pierre Boutroux. Paris, Hachette, 1908, in 3 vols.

historical note before the reimpression of the Traité du triangle arithmétique (volume 3, pages 435–444) never mention the name of one of the greatest European mathematicians of the sixteenth century.

GENOA, ITALY,
June, 1909.

DARWIN'S SCIENTIFIC PAPERS.

Scientific Papers. By SIR GEORGE HOWARD DARWIN, K.C.B., F.R.S., Plumian Professor in the University of Cambridge. Vol. I, *Oceanic Tides and Lunar Disturbance of Gravity*, xiv + 463 pp.; Vol. II, *Tidal Friction and Cosmogony*, xvi + 516 pp. Cambridge University Press, 1907, 8. Royal 8vo.

IF one were in need of an example to illustrate the English use of the term "applied mathematics," it would hardly be possible to find a better one than that furnished by the scientific papers of Sir George Darwin. Many investigations ranging from the purest of pure mathematics to the observational portions of the phenomena of nature have at times been placed under this title in our journals and treatises, and after all our language is governed by general usage and not by arbitrary rules. But in the use of an imported term it would seem better, in spite of the customs of late years, to take the foreign value rather than the domestic as the basis of definition. This definition implies an actual or suggested relation between a problem set forth on arbitrary hypotheses and the observed phenomena of matter in space. The course which the argument follows — the laying down of hypothetical laws approximating as nearly as may be to those of nature, the translation into and development of those laws by means of symbols, and the final transference from the symbols back to the phenomena, is well known. The skill of the applied mathematician is chiefly shown in his management of the second of these stages so as to produce as much as possible in the third.

One cannot read any of Professor Darwin's papers without noting this attitude towards his work, and it is markedly shown in the excellent prefaces with which he introduces the papers contained in the two volumes under consideration. In them he gives a clear synopsis of each memoir, with observations on the results obtained. These observations are almost always on

the physical aspects of the problems he investigates and but little is said of the mathematical methods and the long and laborious developments which were sometimes necessary to arrive at the ends he had in view. Incidentally, it may be mentioned that it is quite possible to obtain a general idea of the contents of the memoirs from the prefaces alone, a more complete summary being given at the end of each of the longer memoirs. An interesting and important part of the preface to volume II is the author's summary of the present attitude of those best capable of judging towards the more speculative portions of his work.

The papers published by Sir George Darwin up to the present time lie chiefly in the domain of hydrodynamics and elasticity. Part I of the first volume consists mainly of memoirs dealing with the practical problem which has as its final object the prediction of the times and heights of the tides at any port. There are two general methods of attacking this question. The first, known as the equilibrium theory, is based on the supposition that the position which the water on the earth has at any instant is the same as that which it would have if the centers of the earth and moon were for that instant in their actual positions but relatively at rest. In other words, in the equations of motion we neglect all the effective forces (mass accelerations), except that due to the rotation of the earth round its axis. Even with this simplification the application of the theory to actual tidal effects is still remote, on account of the considerable effect which the distribution of land and water has on the height and time of the tide at any place. The "corrected equilibrium theory" attempts to take some account of this distribution by the evaluation of certain integrals over the water boundaries. This matter is dealt with numerically by Darwin in conjunction with H. H. Turner in paper 8. The boundaries in general are supposed to follow lines of latitude and longitude, $l = \text{const.}$, $\lambda = \text{const.}$; Professor Turner has obtained a closer approximation by making some of them follow the lines $\pm al \pm b\lambda = \text{const.}$, where a, b are small integers. It would be interesting to compare these results with integrals computed by following Love's harmonic division of the earth's surface.*

The only practical use which is made of the dynamical theory is the discovery of the periods of the principal tides on the sup-

* Presidential address, Section A, B. A. A. S. 1907.

position that the times and heights of the water at any place may be expressed as a sum of harmonic terms, each of which constitutes a "tide." Theoretically, the amplitude and phase of each tide are also determinate but owing to mathematical difficulties, they are always obtained from the observations. The number of terms to be taken into account is of course infinite, but for most places only ten or twelve need be considered, the observations themselves showing that the sum of the remainder may be neglected. Sir George Darwin has succeeded in putting this problem into an economical form. First, the determination of the constants, the same for all time for a particular port, from the observations of the tides for a year or more; and second, the approximate determination of the tide, when these constants and the position of the moon are given, by methods which are within the grasp of the ordinary navigator. More than half the first volume is devoted to these questions.

An interesting point in connection with the terms or tides of long period (a fortnight or a month) is the possibility that they would be considerably affected by an elastic yielding of the solid earth to the tide-generating forces. This was subjected to computation by Sir George Darwin and thence he deduced evidence that the earth's effective rigidity is at least as great as that of steel. The volume closes with the papers which contain an account of the experiments performed by himself and Horace Darwin to measure the lunar disturbance of gravity. This attempt did not meet with success at the time, but quite lately it has been found possible by means of improved apparatus to observe this disturbance.

It is in the second volume that we find the memoirs in which Sir George Darwin has developed the evolution theory of tidal friction which has its chief application in the past history of the earth and moon. In tracing this history back, several stages are to be noted. The first or present stage is that in which we consider an earth practically rigid and more or less covered by water. The attractions of the moon and sun cause tides which give rise to secular effects on account of friction in the motion of the water relative to the earth. The effect is two-fold: it increases the period of rotation of the earth on its axis and also the period of rotation of the moon round the earth. The changes in these two periods are not the same, and the combined effect to an observer situated on the earth is an apparent shortening of the period of rotation of the moon

round the earth. Some other body must be observed in order to separate the two effects: Professor Newcomb had hoped to examine the observations of Jupiter's satellites for this purpose. Since it directly concerns our measure of time, the question is of great importance and it may be advisable to collect observations of every kind which can be affected and examine them with this end in view.

A certain disagreement between observation and gravitational theory in the mean period of the moon's motion has long been known under the name of the "unexplained secular acceleration of the moon's mean motion" and it has most generally been ascribed to this differential effect of tidal friction. For many years its amount was considered to be such as to change the mean longitude by six seconds of arc in one century from the epoch of reckoning. Later, Newcomb's investigations changed it to four and later still to about two seconds. More careful research into the records of ancient eclipses, which form the chief means for obtaining the number, by Newcomb, Cowell, and others have shown how little dependence can be placed on the majority of them, and the number of eclipses to be rejected for this reason has continually increased. It is now doubtful whether we can state that there is any residual effect greater than one second to be ascribed to tidal friction or other unknown cause. This, of course, does not invalidate Darwin's work, since the cause and therefore the effect, though small, is real; it simply alters the time estimates for the total changes. Other effects are changes in the inclination and eccentricity of the lunar orbit.

The second stage is that of a viscous or imperfectly elastic earth in which the main tides are bodily, and it is in this stage that the past history of the earth and moon has been most fully developed by the author. The results are too well known to need restatement here. Volume II contains the memoirs in which they were developed, the paper embodying his remarkable discussion of the series of changes in the orbit of the moon due to the tidal distortion of the earth being that numbered 6. Looking backward in time, we observe the moon brought comparatively near to the earth.

In the third stage the earth is treated as a liquid body in which the effects of viscosity are probably small compared with those of inertia, in contradistinction to the second stage for which the reverse is the case. The problem is now that of a rotating

liquid mass disturbed by a body moving round it and at a distance which may be nearly of the same order of magnitude as the tidal distortion. In the fourth, and in this development the final, stage the two bodies form one liquid rotating mass; the separation is supposed to have been effected by the period of rotation being too short for secular stability. The problem is therefore a consideration of the forms and stabilities of the equilibrum of rotating liquid masses under gravitation. The memoirs dealing with these stages will be given in volume III.

We have then a series of problems the connection between which is continuous from the physical point of view, but for which the mathematical representation is different. These differences of representation may be due to the mathematical difficulties which are presented in an attempt to treat them all as successive stages arising from giving to the time successive values in the same set of formulas, but more probably they are due to the real impossibility of obtaining a single formula adapted to numerical computation for the whole series of changes. We have an analogy in the representation of an analytic function containing singular points in the finite part of the plane; the same power series will not serve for every region of the plane. To continue the analogy, we may say that the different regions for which the formulas are available have been to a certain extent mapped out and the formulas more or less accurately obtained, but the connections between these regions are narrow and the labor of finding values for the function in these connecting portions, so as to make the numerical results continuous, becomes very great. Thus the gaps in Darwin's work which are most serious from the point of view of evolution are those which occur between the successive stages. But though mathematical analysis has not succeeded in bridging them, the crossing is less difficult, if less accurate, by argument from general physical principles.

The method for the solution of these problems adopted by Sir George Darwin is nearly always based on that used to solve the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

When the body approximates to a sphere the potentials are expanded in a series of spherical harmonics and these harmonics, in order to satisfy the hydrodynamical equations, have coefficients

which depend on the time. Much careful consideration is necessary in making the choice of those harmonics which will produce sensible effects. And the time element is an important factor. If the harmonics are periodic with respect to the time, we obtain oscillatory changes the magnitudes of which are directly seen, and they produce no secular changes in the system. But if they are not periodic, the changes produced will vary very considerably with the length of the interval over which they are discussed. We cannot then neglect non-periodic terms even when the coefficients may be small over intervals of time comparable with those of the periods of the larger terms whenever it is desired to discuss secular changes. This is one of the chief pitfalls in many of the problems of celestial mechanics. The necessity for using infinite series of functions the coefficients of which do not follow any simple law carries with it the necessity for stopping the series at some point, and in the great majority of cases we are not able to discover the error committed by so doing. Thus a doubt will continually arise as to whether some neglected term or terms may not considerably modify the results over the chosen interval of time. The doubts may become greater when the constants of the solution have to be determined from observations which should extend far back into the past to obtain the accuracy demanded.

As will be gathered from the above remarks, the arrangement of the papers is mainly by subjects, so that each volume is to a certain extent complete in itself. A chronological list of the papers published to date is given. The third volume containing papers on figures of equilibrium of rotating liquids will embody the investigations which lead to the possibility that the moon separated off from the earth by fission; while the fourth volume will consist of the memoir on periodic orbits, addresses, and miscellaneous papers not included under the preceding titles.

ERNEST W. BROWN.

YALE UNIVERSITY,
July 28, 1909.

SHORTER NOTICES.

Vorlesungen über die Elemente der Differential- und Integralrechnungen und ihre Anwendungen zur Beschreibung von Naturerscheinungen. Von HEINRICH BURKHARDT. Leipzig, Teubner, 1907. 8vo. xi + 252 pp.

THIS volume has been prepared to meet the needs of a growing body of students who are finding the calculus useful in certain sciences which do not call for its fuller development—notably chemistry, mineralogy, and statistics. The author has aimed so to choose the material for exposition that the volume may serve as a first course in the calculus for others who, from choice or necessity, require an advanced course. The problem is complicated by seeking to avoid all "arithmetization" of material such as would be necessary for a complete and rigorous treatment of the fundamental limit

$$\lim_{n=\infty}\left(1+\frac{1}{n}\right)^{n}.$$

This limit is, in a sense, the pons asinorum of the calculus and Professor Burkhardt feels that the chemist can not be gotten over it easily, and that even the mathematician finds difficulty with it when he is introduced to it too abruptly and not in connection with a complete discussion of the concept of limit. (Preface, page v.) The avoidance of arithmetic necessitates, from the author's point of view, leaving out all reference to convergence and to infinite series. Thus the usual series are spoken of as formulas and treated as approximations to the value of a function for certain of its arguments.

It is certainly interesting and instructive to any teacher of mathematics to know how a mathematician of Professor Burkhardt's attainments seeks to solve the problem he has set for himself.

The book is divided into ten chapters and a Nachtrag. It has a good table of contents and a complete register and is put together in the well known style of the publishers. It begins by a long, but in no wise uninteresting, introduction leading up, by easy stages through the laws of motion, to the notion of differential quotient. In this first chapter one finds the

usual concepts and symbols together with an exposé of the fundamental principles of cartesian geometry as applied to the straight line and to tangents in general.

The second chapter is devoted to the differentiation of rational functions, one section being set aside for the special consideration of the linear fractional function. This enables the author to illustrate the notion of infinitely great and to introduce the hyperbola and its asymptotes.

The differentiation of irrational functions is considered in the third chapter. The rule for differentiating inverse functions is first taken up and, by its aid, the rule for differentiating x^m is shown to hold for any rational m. The chapter closes with the rule for differentiating a function of a function.

The student is now able to differentiate any algebraic function, and in the next chapter he is introduced to the inverse operation. The chapter opens by finding a function whose differential quotient is gt, without previous knowledge of the differential calculus. This is the nearest approach to the treatment of a definite integral as the limit of a sum. Such a treatment would, of course, be impossible from the author's point of view. Integration is defined as the inverse of differentiation, and the integral of x^m is found for all values of m except $m = -1$. We find here the geometrical interpretation of the definite integral as the area beneath the curve whose ordinates, between the limits, are given by the integrand. The trapezoidal formula for calculating approximately the value of a definite integral is followed by its application to the integral

$$\int_1^2 \frac{dx}{x}.$$

Chapter five contains the crux of the whole volume. Professor Burkhardt takes his dilemma by both horns, so to speak, and *defines*

$$\int_1^x \frac{d\xi}{\xi}$$

to be the natural logarithm of x. By means of the geometrical interpretation just given, the student is shown that this new function is defined for all positive values of x and that it is not defined for $x = 0$. The question of its definition for negative values of x is expressly excluded.

The addition theorem for logarithms follows easily from the definition, and a comparison with common or Brigg's logarithms leads to the formula

$$\log x = \frac{\log z}{\log \text{Brigg } z} \cdot \log \text{Brigg } x.$$

This formula enables one to calculate the natural logarithms of all positive numbers, with the help of a table of common logarithms, as soon as the natural logarithm of *any* number $z \neq 1$ is known. The natural logarithm of 2 was found approximately in the preceding chapter. Thus we are across the bridge and the student has not realized its existence, the approaches have been so cunningly concealed!

The reviewer is reminded of the proof for differentiating the logarithm given in Olney's Calculus (1871) and there credited to the astronomer, J. C. Watson. Olney says of this proof that it "banishes from the calculus the last necessity for resort to series to establish any of its fundamental operations." But the modulus of the common system — there's the rub! Olney resorts to infinite series to find it and Burkhardt relies upon the trapezoidal formula, which is dangerously near to the same thing. The fact is, it seems a doubtful policy to avoid all reference to arithmetic and substitute therefor the notion of an integral as a function of its upper limit. The student must feel that the method is, to say the least, round about. The question is pedagogically difficult, but teachers will find greater satisfaction in treating it as is done, for example, in Osgood's Calculus even for the student of chemistry or statistics.

The remainder of the chapter is devoted to the integration of rational fractional functions but only for the cases where the denominator has distinct or multiple real roots. Application is made to the integration of simple ordinary differential equations expressing chemical reaction between two or more substances.

The sixth chapter is the longest in the book and in many respects the most interesting. It contains sections upon higher derivatives, the mean value theorem, Maclaurin's and Taylor's formulas. The three important special cases, the binomial formula, the exponential formula, and the logarithmic or Mercator formula, are emphasized and illustrated. Numerical calculation is given a prominent place, and sections on reckoning with small quantities, on approximate solution of nu-

merical equations, and reversion of approximate formulas are given. Section 50, on the generalized mean value theorem and the remainder in the Maclaurin formula, is the nearest approach to a complete treatment of the development of a function in series which Professor Burkhardt allows himself. Maxima and minima values of a function receive only the barest treatment and in only the simplest cases. The same thing is true of the evaluation of indeterminate forms. One may, of course, disagree with Professor Burkhardt's placing of emphasis upon the materials chosen for this chapter. It would seem that maxima and minima values of a function were of sufficient importance, even to the chemist, to deserve fuller notice. But one must agree that the chapter is charmingly written and will commend itself to teachers for its simple straightforward presentation of subjects which many students find tedious and difficult in their formal setting.

Interpolation forms the subject of the seventh chapter. It is first performed by setting up a rational integral function to fit the observations, and a problem is fully worked out with all the necessary numerical calculation of coefficients. In the next place it is performed by finite differences, and lastly where the number of observations is greater than the number of constants to be determined.

Some of the numerical calculation might well have been left to the student and a paragraph or two devoted to logarithmic plotting, since it is just as easy to fit the curve $y = bx^a$ to the observations by plotting the logarithms of x and y and then using the straight edge as it is to fit $y = ax + b$ to the observations in the first place (page 184), and it may turn out to fit more accurately.

Chapters eight and nine are each but twelve pages in length. The first treats of the infinitesimal or differential, and the second of functions of two variables and partial derivatives.

In the tenth chapter, the student is led to see the use of trigonometric functions in describing periodic phenomena. Rules for differentiating and integrating trigonometric functions follow, and by inversion for the circular or *cyclometric* functions. Trigonometric interpolation is considered, as well as three simple cases of vibration, viz., free vibration, damped vibration, and forced vibration, in case the external force is itself periodic.

The Nachtrag is devoted to interpolation by means of exponential functions.

The book, as a whole, commends itself for its simplicity of presentation. The treatment of the logarithm is a doubtful pedagogical expedient but there is no lax rigor about it. The responsibility is shifted to the numerical calculation of logarithms, just as was done in Olney's Calculus over thirty years ago.* A student who has had elementary training in algebra and trigonometry can read the book without difficulty and, in the main, it presents enough of the calculus and its applications to serve that body of students of which we have spoken at the beginning. But it does not represent what we, in America, have come to consider as a first course in the calculus.

L. WAYLAND DOWLING.

Vorlesungen über die Weierstrasssche Theorie der irrationalen Zahlen. Von VICTOR VON DANTSCHER. Leipzig und Berlin, Teubner, 1908. vi + 79 pp.

IN the preface we are told that these lectures are based upon a course given by Weierstrass in the summer semester of 1872, which was followed by the author of the present volume, and upon an elaboration of a later course given in 1884. The work under review is, however, not a mere reproduction of things given by Weierstrass, but it is the direct outcome of a course given repeatedly at the University of Gratz by Professor von Dantscher. It furnishes an easy and clear introduction to that theory of irrational numbers which was first developed by Weierstrass in his lectures at the University of Berlin, and it has decided pedagogic as well as scientific value.

C. Méray was the first to give a purely arithmetic meaning to the term irrational number,† and the theories developed by him, G. Cantor, Heine, and Dedekind have perhaps become better known than the theory of Weierstrass. This may be partly due to the fact that no expository publication relating to this theory was ever prepared under the direction of Weierstrass, and only the fundamental elements of this theory have been accessible in the works of Kossak, Pincherle, Biermann, and others. The first of these was based upon a course of lectures given by Weierstrass during the winter semester of 1865–6, and it was published in 1872 under the title "Die Elemente

* It should be stated that Olney sought only to avoid infinite series. The Watson proof of the rule for differentiating the logarithm tacitly assumes that $dx^n/dx = nx^{n-1}$ holds for all real values of n.

† Encyclopédie des Sciences mathématiques, tome I, vol. 1, p. 149.

der Arithmetik, Programm-Abhandlung des Werder'schen Gymnasiums."

The subject matter of the present volume may be sketched as follows: After a brief historical introduction and a proof of the fact that it is impossible, in the domain of rational numbers, to extract the nth root of a natural number which is not an nth power of a natural number, the fundamental concept "additive aggregate of an infinite number of positive rational numbers" is introduced, and we are reminded that the terms "additive" and "multiplicative," as applied to aggregates, are not due to Weierstrass. It seems desirable to use the term aggregate instead of the term set (Menge, ensemble) since a set of numbers generally implies that the numbers considered are discrete, while this is not commonly the case as regards the aggregates of numbers used by Weierstrass.* The term "additive" as applied to an aggregate of an infinite number of positive rational numbers implies that every possible sum of a finite number of the "members" of the aggregate is to be formed.

To make it possible to operate with these aggregates it is necessary to give a definition for the equality of two aggregates. In framing such a definition the term part (Bestandteil, partie) of a positive rational number a is used for every positive rational number smaller than a. Hence two rational positive numbers are equal if every part of each is a part of the other, and, similarly, two additive aggregates are said to be equal whenever every part of each is a part of the other. As there are additive aggregates whose equality may be established according to this definition irrespective of whether one of these aggregates is increased or diminished by any finite number, the author restricts himself to the consideration of additive aggregates which are such that every part is less than a given finite number. These aggregates are said to be convergent while all other aggregates are said to be divergent.

The explanation of these fundamental notions is followed by the representation of a convergent additive aggregate as a systematic fraction, and the proof that these aggregates may be combined by addition, subtraction, multiplication, and division after the definitions of these operations have been properly formulated. In view of the fact that the ordinary rules of arithmetic apply to these aggregates, they are called numbers, and the domain of rational numbers is extended by adding to

* J. Tannery, *Bulletin des Sciences math.*, vol. 32 (1908), p. 103.

it these additive convergent aggregates, which may also be called irrational numbers. It is then proved that, in this enlarged domain, it is possible to extract the nth root of any natural number, and to represent the ratio of two lines, whether they are commensurable or incommensurable.

The subject matter of the rest of the volume may perhaps be sufficiently evident from the general headings of the last three lectures. They read as follows: Additive aggregates of an infinity of positive and negative rational numbers; additive aggregates of an infinity of complex numbers of the form $a + bi$; multiplicative aggregates of an infinity of numbers. The value of the volume is greatly enhanced by illustrative examples, and it may be heartily recommended even to those who are just beginning graduate work in our universities. It need scarcely be added that a clear comprehension of this theory of irrational numbers will clear up many difficulties as regards the theory of absolutely convergent series with numerical terms.

G. A. MILLER.

Magic Squares and Cubes. By W. S. ANDREWS. With Chapters by PAUL CARUS, L. S. FRIERSON, C. A. BROWNE, JR., and an Introduction by PAUL CARUS: Chicago, The Open Court Publishing Company, 1908. vi + 199 pp.

Among the Arabians magic squares were known in the ninth century of our era and about this time they played an important rôle in Arabian astrology. A special work on the subject is attributed to an Arabian mathematician named Tâbit ben Korrah who died in 901,* and H. Suter mentions several other early Arabian writers on this subject in his work entitled Die Mathematiker und Astronomen der Araber und ihre Werke. These facts are not in accord with the statement on page 1 of the book under review, which reads as follows: "The earliest record of a magic square is found in Chinese literature dated about 1125 A. D."

The present work is, in the main, a direct reprint of articles which appeared in the *Monist* during recent years. Its author is an electrical engineer who, during his leisure hours, "has given considerable thought to the working out in his own original way the construction of magic squares and cubes of various styles and sizes." As may be inferred from this excerpt

* Encyclopédie des Sciences mathématiques, t. 1, vol. 3 (1906), p. 63.

from the announcement and also from the preceding paragraph, the author made little use of the extensive literature on the subject, but has aimed to give a clear and interesting account of the results to which his own labors and those of his correspondents have led. An important exception is furnished by the chapter on the "Franklin squares," which gives a very interesting account of magic squares constructed by Benjamin Franklin, including a letter in which Franklin says "I make no question, but you will readily allow the square of 16 to be the most magically magical of any magic square ever made by any magician."

The general headings of the various parts of the book are as follows: Introduction by Paul Carus, magic squares, magic cubes, the Franklin squares, reflections on magic squares by Paul Carus, a mathematical study of magic squares by L. S. Frierson, magic squares and Pythagorean numbers by C. A. Browne, some curious magic squares and combinations, notes on the various constructive plans by which magic squares may be classified, and the mathematical value of magic squares. Under the sub-heading "Mr. Browne's square and lusus numerorum" Paul Carus gives instances of numbers which exhibit surprising qualities without being in the form of a magic square.

In 1896 Emory McClintock read a paper before the American Mathematical Society entitled "On the most perfect forms of magic squares with methods of their construction" in which he introduced the term pandiagonal magic squares for a type of squares which were called diabolic by Lucas and are generally known in Europe by the latter term. The paper by McClintock was published in the *American Journal of Mathematics*, volume 19, and constitutes one of the most important American contributions to the subject. Extensive bibliographical data on this subject may be found in volume 1 of the Subject Index of the Royal Society of London Catalogue of Scientific Papers, pages 84 and 85; in Ahrens's "Mathematische Unterhaltung und Spiele," 1901; and in the "Encyclopédie des Sciences mathématiques," tome 1, volume 3, pages 62 to 75. It is of interest to observe that the French edition of the great mathematical encyclopedia devotes thirteen pages to this subject while less than a page is devoted to it in the German edition.

As has been observed above, the few historical references in the present work should not be taken seriously, but in other

respects it can be commended highly to those who are attracted by marvellous relations among natural numbers. The author is looking forward to a second edition in which a number of slight errors will be corrected, and he has had the courtesy to send the reviewer a marked copy, in which the following changes are suggested: The term "perfect square" as used on page 2 is replaced by "regular square." In the second and third lines from the bottom of page 5 "twenty-eight" and "sixteen" are replaced by twenty and eight respectively, and in the first line of page 6 "twelve" is replaced by sixteen. The term "prime number" as used on page 14 and in many other places in the book is replaced by primary number. In the last line on page 65 the expression "first and last" should read last and first. Near the middle of page 179 the statement marked I. should be followed by "with four exceptions." These errors are, however, not sufficiently serious to detract much from the value of the volume.

G. A. MILLER.

Exercices et Leçons d'Analyse. By R. D'ADHÉMAR. Paris, Gauthier Villars, 1908. 208 pp.

THE subtitle of this volume is "Quadratures, équations différentielles, équations intégrales de M. Fredholm et de M. Volterra, équations aux dérivées partielles du second ordre." It will be seen that the topics treated are thoroughly up to date. The book is meant, as the author says in his preface, to supplement the larger Traités and Cours. An introduction of 22 pages presents a brief statement of some of the principal theorems on differential geometry and analysis, together with references for their proofs and for further developments. Then follow chapters on quadratures; the functions of Legendre, Bessel, Euler, etc.; partial differential equations of the elliptic type, including a brief treatment of Fredholm's integral equation; equations of the hyperbolic and parabolic types; and two chapters on miscellaneous problems. The book, in spite of its decidedly fragmentary character, will prove useful both by furnishing a source of interesting problems and by giving the reader at least a superficial idea of many recent developments in analysis. The indications given as to the scope and purport of theorems mentioned (to say nothing of their proofs) are, however, frequently so meagre that the reader seeking to gain information will often be in doubt as to how they should be

interpreted. Occasionally an actually misleading statement occurs, as when on page 116 in discussing the essential difference between the Cauchy-Kowalewski existence theorem for Laplace's equation and the "problem of Dirichlet" it is at least strongly implied that the former theorem does not apply to closed curves, whereas it applies exactly as well to closed curves as to open ones, but in both cases, and this is the essential point, to only a small neighborhood of the curve.

Judiciously used by a person who is able to perceive that he does not understand a thing when that is the actual case, the book will prove a source of inspiration.

MAXIME BÔCHER.

Analytische Geometrie auf der Kugel. Von Dr. RICHARD HEGER. Sammlung Schubert, LIV. Leipzig, G. T. Göschen, 1908. 12 mo. vii + 152 pp.

THERE is a spherical trigonometry; why not also a spherical analytical geometry? This question interested mathematicians towards the end of the eighteenth century and the beginning of the nineteenth, and there resulted numerous papers published in the periodicals of the time. Certain problems had been solved at an earlier date and others have appeared up to within a decade ago. It is the object of the author of the little volume before us to bring this material together in a convenient form and to arrange it along the lines of the usual text upon plane analytic geometry. The book begins by explaining several coordinate systems upon the sphere. Dr. Heger adopts that one in which the homogeneous coordinates of a point are the sines of the angles whose arcs are drawn perpendicularly from the point to the sides of the spherical triangle of reference. This triangle of reference is assumed to be trirectangular. The homogeneous coordinates of a great circle are taken to be the point coordinates of one of its poles. Many of the formulas are exactly the same as the analogous formulas in plane geometry. For instance, the necessary and sufficient condition that a point (x, y, z) lie upon a great circle (u, v, w) is

$$ux + vy + wz = 0.$$

Small circles and conics in general are represented by quadratic equations. There is a theory of poles and polars and of tangents, all of which is analogous to plane analytic geometry.

The book closes with a section devoted to spherical cubics. There are 152 pages, a bibliography, and a short index.

The book will find its greatest use in technical schools. Spherical trigonometry has come to be counted as one of the technical studies and taught in connection with geodetic surveying or with astronomy. It is the same with spherical analytic geometry from Dr. Heger's point of view, and the field of application is narrower.

Geometry upon the sphere is most interesting when studied as a correspondence between the sphere and some other surface — in particular, the plane. This point of view is hinted at in section 8 in explaining Gudermann's axial coordinates, but no general theory of correspondence between sphere and plane is worked out.

Dr. Heger's analytical geometry amounts to a correspondence between the sphere and the projective plane. There is another geometry upon the sphere arising from a one to one correspondence with the plane whose results are quite as useful, in their way, but which does not come within Dr. Heger's field of view.

L. WAYLAND DOWLING.

Die Elemente der Mathematik. Von ÉMILE BOREL, Professor an der Sorbonne zu Paris. Vom Verfasser genehmigte deutsche Ausgabe, besorgt von PAUL STÄCKEL, Professor zu Karlsruhe i. B. Erster Band: *Arithmetik und Algebra.* Mit 57 Textfiguren und 3 Tafeln. Leipzig und Berlin, B. G. Teubner, 1908. xvi + 431 pp.

This work is a German translation, or rather a "Bearbeitung," in one volume, of the three French booklets published by Borel in 1903. Borel traverses in his texts the ground to be covered in arithmetic and algebra by pupils between the ages of 14 and 17, in accordance with the courses of study laid out officially in 1902. The distinctive feature of this movement lies in the emphasis laid on graphic work, on the concept of a variable and of a function. Stäckel says in his preface to the German edition that, in view of the wide divergence of opinion as to what can be accomplished in this line with elementary pupils, the only way of arriving at an understanding and thereby at an actual realization of the contemplated reform, appears to be in showing by an example just what that reform really aims to achieve and how the subject can be developed

with pupils of the ages named. Such an example is furnished in Borel's texts. As Stäckel remarks, this publication is intended only for teachers. Since the reform movement in France and Germany is essentially the same as in the United States, the book under review, coming from authors of distinction, cannot fail to be of interest to American readers.

FLORIAN CAJORI.

A Treatise on the Mathematical Theory of Elasticity. By A. E. H. LOVE, M.A., D.Sc., F.R.S. Second edition. Cambridge University Press, 1906. xvii + 551 pp.

THE first edition of this important work was published in two volumes in 1892 and 1893. The present edition is a new treatise which contains some extracts from the old one. The object of the book is threefold, namely, to be useful to engineers, to set forth the physical notions and analytical processes which are also used in other branches of physics, and to afford a complete picture of the present state of the science of elasticity. The book commences with an excellent historical introduction which explains the parts taken by various eminent mathematicians in establishing the theory. The first four chapters are concerned with the analysis of strain and stress, the equations of equilibrium and small motion, the expression of the stresses as functions of the strains, and the connection between the mathematical theory and technical mechanics. Chapter V opens with a useful recapitulation of the essential parts of the preceding chapters and the author proceeds to illustrate them by a number of simple examples which are needed for the subsequent development of the subject. In Chapter VI there is a discussion of the elastic constants. The 6 components of stress are linear functions of the 6 components of strain and hence depend on 36 constants. The law of conservation of energy reduces the number of constants to 21. The hypotheses of Navier and Cauchy concerning the constitution of matter (according to which bodies are regarded as made up of material points which are supposed to act on each other so that the mutual action between each pair of points is along the line joining them and is a function of the length of the line) leads to 6 relations which are called Cauchy's relations, so that the number of constants is reduced to 15. As the author remarks in the historical introduction, our views concerning the constitution of matter have changed so that the

argument as to whether the number of constants is 21 or 15 holds a subordinate position. The development of the atomic theory in chemistry, of statistical molecular theory in physics, and the discovery of electrical radiation, have shaken our confidence in the hypothesis of central forces between material points. In the case of crystals possessing certain kinds of symmetry the number of constants may have various values from 21 down to 2 for an isotropic body. Chapter VII discusses the uniqueness of a solution and also Betti's reciprocal theorem, which subsequently proves to be of great importance in developing a general method of solution. The various singular solutions analogous to $1/r$ in ordinary potential theory are next obtained, and an interesting application follows, namely, Hertz's theory of the distribution of pressure between two bodies in contact. Chapter X opens with an excellent resumé of the theory of potential which explains the two lines of attack, namely, the method of a series of harmonic functions and the method of singularities or the use of Green's functions. The author here states that little progress has been made with the existence theorem of elasticity. Shortly after the present work appeared however, Tedone published in the Encyklopädie der mathematischen Wissenschaften a valuable account of the memoirs dealing with the existence theorem and remarked on the probability of using Fredholm's integral equations. Lauricella and others have since successfully applied integral equations to the problem and the new book on the Equations of Fredholm by d'Adhémard contains an account of these researches. Professor Love proceeds to give a full account of Betti's method of integration and of Cerruti's application to the problem in which the boundary is a plane. In the next chapter the problem of the sphere is discussed by means of harmonic functions. Tedone has solved the same problem by the use of Green's functions. After discussing the vibrations of solids, the author takes up the important practical problems of the torsion and flexure of beams and regrets the slowness of engineers in adopting the exact methods of solution. The sections of a beam do not usually remain plane, with the curious result that the greatest stress is not at the point furthest from the center. The remainder of the book is devoted to the theory of deformable bodies and is substantially identical with the second volume of the first edition. In the case of thin rods the theory is fairly simple, although a

long analysis is required to justify the approximation that the bending couple is proportional to the curvature. In the cases of plates and shells the necessary analysis is still more lengthy. The problem is complicated by the fact that the bending is usually accompanied by stretching particularly near the edges. The book concludes with an account of the important practical problem of the stability of cylindrical shells. It is perhaps needless to say that the treatise can be heartily commended both as a text-book and a book of reference. A German edition was published by Teubner in 1907.

F. R. SHARPE.

Les Découvertes modernes en Physique. Par O. MANVILLE. 2ème édition. Paris, A. Hermann et Fils, 1909. 463 pp.

ONLY a year after the first edition of Manville's short book of 182 pages on Les découvertes modernes en physique a second edition was needed. The author evidently did not have to contend with costly electrotype plates in which the publisher would allow few changes, for he has practically written a new book about three times the size of the first — the term second edition is really a misnomer. The new work is divided into two parts, entitled Electricité et matière and Les ions et les électrons dans la théorie des phénomènes physiques — La matière et l'éther. This entire rewriting and expansion of the original is very fortunate. The state of fundamental electrical theory is to a considerable extent still speculative, and experiments which reveal new and sometimes nearly crucial results are still of frequent occurrence. To write at all on this subject brings with it the liability and desirability of rewriting after the lapse of a very short period.

From the title of the work we might be inclined to fear that the author had written a popular and unreliable essay on the wonders of recent discoveries. Fortunately this is by no means the case; many chapters contain considerable hard physics and more or less hard mathematics, which require and repay close application on the part of the reader. The presentation, however, let it be stated, deals with a vast variety of interrelated physical data after the manner of the experimental physicists rather than with the broad mathematical groundwork of electrical theories as treated by such theorists as Larmor, Lorentz, or Minkowski. Sooner or later, theory and experiment in regard to atomic electricity will probably be well knit together;

at present the data are too diverse and too numerous to be worked into any mathematical theory, and the different theories are in need of crucial experiments. In addition to these general observations only a few words as to the contents of the work under review need be added.

The author begins with the discussion of the conduction of electricity by liquids and of the ionic theory of electrolysis. He then takes up the corresponding questions for gases; this, of course, requires a much longer treatment, as it involves cathode rays, X-rays, Lénard rays, and electrons. Then follows the electronic theory of matter with a detailed discussion of radioactivity. This ends the first part of the work and is to all intents and purposes merely a new edition of the original volume. In the second part the author goes over the data above presented and applies the facts and points of view acquired to the connected study of the phenomena involved — first, as regards liquids, second, as regards gases, whether ionized or not, and third, as regards solids. The volume finally closes with a discussion of matter and ether.

To write the second part must have been a much harder task than to write the first; for whereas the fundamental facts as to atomic electricity are tolerably firmly established and generally agreed to, the manner in which those facts shall be interpreted and combined into physical theories is by no means so well settled. For instance in the treatment of electric and thermic conductibility in metals there is the theory of J. J. Thomson and the English school of physicists, that of Drude and the German school, and that of Lorentz. None of these theories is as yet entirely satisfactory; the author carefully presents them all and makes a few comments as to their several defects or excellencies. It will thus be seen that we are in possession of a work which will serve probably better than any other one book to orient the student in regard to the modern physics of electricity and to place him in a position to read critically the original memoirs that have appeared or may appear — in short, to put him quickly and easily where he can and must do some real thinking for himself.

E. B. Wilson.

NOTES.

THE sixteenth annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Boston in convocation week, in affiliation with the sixty-second meeting of the American Association for the advancement of science. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary by December 15. Abstracts intended to be printed in the announcement of the meeting must be submitted by December 8.

THE closing (October) number of volume 10 of the *Transactions of the American Mathematical Society* contains the following papers: "The summability of the developments in Bessel functions, with applications," by C. N. MOORE; "Singular points of ordinary linear differential equations," by G. D. BIRKHOFF; "Automorphisms of order two," by G. A. MILLER; "Resolution into involutory substitutions of the transformations of a non-singular bilinear form into itself," by DUNHAM JACKSON; "On singular points in the approximate development of the perturbative function," by F. W. REED. Also "Notes and errata," volumes 8 and 10; Table of contents, volume 10; and Indices by authors and subject matter of volumes 1–10.

THE opening (October) number of volume 11 of the *Annals of Mathematics* contains: "Theory of floating tubes," by FRANK GILMAN; "Generalized geometric means and algebraic equations," by OTTO DUNKEL; "The geometry of chains on a complex line," by J. W. YOUNG.

THE publishing house of Ginn and Company, New York, announces that the following books are in preparation: "Theoretical mechanics," by P. F. SMITH and W. R. LONGLEY; "Projective geometry," by O. VEBLEN and J. W. YOUNG. The treatise on differential geometry by L. P. EISENHART has recently been published.

IN the press of B. G. Teubner are the following books on mathematics: Encyklopädie, parts of II 2, III 1, IV 1, IV 2, V 2, V 3, VI 1, and VI 2; Encyclopédie, parts of I 1–4, II 2–5, III 1, IV 2, IV 4; "Verzeichnis der Schriften Leonhard Eulers," by G. ENESTRÖM (*Jahresbericht*, Ergänzungsband III); "Didaktik des mathematischen Unterrichts," by A. HÖFLER; "Niedere Zahlentheorie, Teil II," by P. BACH-

MANN; "Elemente der Geometrie," by H. THIEME; "Lehre von den geometrischen Verwandtschaften, Band IV," by R. STURM; "Synthetische Theorie der Cliffordschen Parallelen und der linearen Linienörter des elliptischen Raumes," by W. VOGT; "Analytische Geometrie der Punktpaare, Kegelschnitte und Flächen zweiter Ordnung," by O. STAUDE; "Funktionentafeln mit Formeln und Kurven," by E. JAHNKE; "Vektoranalysis und ihre Anwendung in der theoretischen Physik," by W. v. IGNATOWSKY.

THE annual list of American doctorates published in SCIENCE for the academic year 1908–1909 contains 378 names, of which 189 are credited to the sciences. The following 15 successful candidates offered mathematics as major subject (the titles of the theses are appended): H. E. BUCHANAN, Chicago, "Periodic oscillations of three finite masses about the Lagrangian circular solutions"; T. BUCK, Chicago, "Oscillating satellites near the Lagrangian equatorial triangle points"; H. T. BURGESS, Yale, "Point-circle correlations"; J. R. CONNER, Johns Hopkins, "Basic systems of rational norm-curves"; L. S. DEDERICK, Harvard, "Certain singularities of transformations of two real variables"; A. DRESDEN, Chicago, "The second derivatives of the extremal integral"; G. F. GUNDELFINGER, Yale, "On the geometry of line elements in the plane with reference to osculating vertical parabolas and circles"; G. W. HARTWELL, Columbia, "Plane fields of force invariant under projective transformations"; D. D. LEIB, Johns Hopkins, "On a complete system of invariants of two triangles"; J. V. MCKELVEY, Cornell, "The groups of birational transformations of algebraic curves of genus 5"; W. D. MACMILLAN, Chicago, "Periodic orbits about an oblate spheroid"; E. H. TAYLOR, Harvard, "On some problems in conformal mapping"; H. I. THOMSEN, Johns Hopkins, "Some facts in regard to plane rational curves"; M. O. TRIPP, Columbia, "Groups of order p^3q^2"; Miss M. S. WALKER, Yale, "A generalized definition of an improper multiple integral."

The number of American doctorates in mathematics for each of the last eleven years is 13, 11, 18, 8, 7, 14, 21, 11, 13, 22, 15.

ANOTHER professorship of mathematics has just been created at the University of Paris, with the designation of chair of the theory of functions. Professor E. BOREL has accepted the

new position. There are now twelve full professorships of mathematics at the Sorbonne, the present occupants being Professors Andoyer, Appell, Borel, Boussinesq, Darboux, Goursat, Koenigs, Painlevé, Picard, Poincaré, Raffy, Tannery. There are also two associate professors, Hadamard and Puiseux, and one chargé de conférences, Dr. Blutel.

THE following advanced courses in mathematics are announced for the academic year 1909–1910:

CAMBRIDGE UNIVERSITY. (Michaelmas term begins October 14, Lent term January 17, and Easter term April 25.) — By Professor A. R. FORSYTH: Differential equations, three hours (M); Differential geometry, three hours (M); Functions of two or more complex variables, three hours (L). — By Professor Sir G. H. DARWIN: Lunar theory, three hours (M); Figure of the earth, three hours (L). — By Professor Sir R. S. BALL: Mathematical astronomy, three hours (M); Applications of geometry to dynamics, three hours (L). — By Dr. E. W. HOBSON: Harmonic analyses, three hours (M); Higher dynamics, three hours (L); Lebesgue integrals, three hours (E). — By Dr. H. F. BAKER: Theory of functions, three hours (M, L); Curves, three hours (M, E). — By Mr. B. A. HERMAN: Hydromechanics, three hours (M, L). — By Mr. H. W. RICHMOND: Algebraic and synthetic geometry, three hours (M, L, E). — By Dr. A. N. WHITEHEAD: Synthetic geometry, three hours (M); Philosophy of mathematics, three hours (L, E). — By Dr. T. J. I'A BROMWICH: Theory of potential and applications, three hours (M, L). — By Mr. A. BERRY: Elliptic functions three hours (L, E). — By Dr. E. W. BARNES: Linear differential equations, three hours (L). — By Mr. C. T. BENNETT: Line geometry, three hours (L). — By Mr. A. MUNRO: Hydrodynamics and sound, three hours (M). — By Mr. G. J. LEATHEM: Elementary electron-theory, three hours (M). — By Mr. J. H. GRACE: Theory of numbers and invariants, three hours (M, L). — By Mr. G. H. HARDY: Integral equations, three hours (E).

AT the session of the Paris academy of sciences held on July 19, 1909, the following prizes were awarded: Binoux prize (fr. 2000) to Professor P. DUHEM for his contributions to the history of science; Pierron-Perrin prize (fr. 5000) to Professor E. MERCADIER, of the Ecole polytechnique, for his work on

acoustics, elasticity and telegraphy; Montyon prize (fr. 700) to M. DE SPARRE, for his work on rational mechanics; Boileau prize (fr. 1300) to Professor BOULANGER, of the University of Lille, for his treatise on general hydraulics; Lalande prize (fr. 540) to M. BORELLY, of the observatory of Marseilles, for his discoveries of small planets and comets; Vals prize (fr. 460) to M. de la BAUME-PLUVINEL for his contributions to astronomy; Pontécoulant prize (fr. 700) to Professor E. W. BROWN, of Yale University, for his researches relative to the theory of the moon.

AMONG the recent contributions to the Euler fund, we mention the subscription for forty sets of the complete works by the academies of science of Berlin and of St. Petersburg, and a cash subsidy of five thousand francs by the latter; a gift of eight thousand francs by the city of Zürich, and the appropriation of five thousand francs by the American Mathematical Society. At the beginning of September the number of sets subscribed for was 275. In view of these facts, the Swiss scientific society, at its ninety-second annual meeting at Lausanne, September 5–8, 1909, unanimously voted to proceed with the publication of the complete works of Leonhard Euler. The preparation of the different volumes will be assigned to one or more specialists in the respective fields; about twenty have already signified their willingness to serve. A general committee of three will direct the whole undertaking.

THE French association for the advancement of science held its thirty-eighth annual meeting at Lille, August 2–7, under the presidency of Professor Landouzy, dean of the faculty of medicine of the University of Paris, with Professor E. LEBON, chairman of the section of mathematics and astronomy. The following papers were read: "Biography of Poincaré," by E. LEBON; "Modular geometry," by G. TARRY; "Fredholm's equations," by H. POINCARÉ; "Instruction in mathematics," by C. A. LAISANT; "Geometric representation," by N. CHAPELON; "Fermat's theorem," by A. GÉRARDIN; "Plane representation of paraboloids," by F. MICHEL; "Hyperelliptic surfaces," by E. TRAYNART; "Theory of integrals of algebraic functions of several variables," by L. AMOROSO; "The method of indeterminants and the solution of functional equations," by M. CARLIER; "On the theory of curves," by A. PELLET; and twenty papers on astronomy. The next meet-

ing of the association will be held at Toulouse during the summer of 1910, under the presidency of Professor C. M. GARIEL, of the University of Paris, the chairman of the section of mathematics being M. BELOT.

THE third annual meeting of the Italian association for the advancement of science was held at Padua, September 20 to 25, under the presidency of Professor V. VOLTERRA. A notable feature of the meeting was the prominence given to general papers read before joint sections, among which was the address given by Professor F. SEVERI, on "Hypotheses and reality in geometric science." The following papers were read before the section of pure and applied mathematics. By A. ALESSIO, "On the reduction of observations for the determination of relative gravity to a rigid bearing of the pendulum"; by U. CISOTTI, "Maxwell's stresses and elastic media"; by A. CROCCO, "Aërial navigation"; by G. GALLUCCI, "On irregular configurations N_3"; by T. LEVI-CIVITA, M. ABRAHAM and M. O. CORBINO, "On the constitution of electric radiation"; by E. PASCAL, "Mechanical integration of differential equations"; by G. RICCI, "On the determination of three-dimensional varieties having prescribed intrinsic properties."

THE Italian mathematical society Mathesis, recently reorganized to consider both scientific and pedagogic questions, held its second annual meeting at Padua, September 20–23, under the presidency of Professor F. SEVERI. The opening address was given by Professor G. LORIA on the existing crisis in the secondary schools. After a symposium on proposed reforms and on more systematic preparation of teachers in which a large number participated, Professor BERZOLARI and BONOLA proposed the organization and publication of an encyclopedia of elementary mathematics in Italian. Professor CASTELNUOVO reported on the work of the international commission on mathematical instruction. The society's prize of 900 francs for the best essay on some elementary subject was awarded to M. PADOA for his essay on the theory of fractions.

PROFESSOR J. PETERSEN, of the University of Copenhagen, has retired from active teaching. Professor N. NIELSEN has been promoted to a full professorship and to be head of the department of mathematics.

DR. H. HAHN, of the University of Vienna, has been appointed associate professor of mathematics at the University of Czernowitz.

PROFESSOR E. NAETSCH, of the technical school at Dresden, has been promoted to a full professorship of mathematics.

THE gold medal of the French association for the advancement of science, for the present year, has been awarded to Professor H. POINCARÉ, of the University of Paris.

DR. J. O. MÜLLER has been appointed docent in mathematics at the University of Bonn.

AT the University of Cambridge Dr. T. J. I'A. BROMWICH has been appointed university lecturer in mathemathics.

PROFESSOR G. A. GIBSON, of the Glasgow and West of Scotland Technical College, has accepted the professorship of mathematics at the University of Glasgow.

PROFESSOR G. A. MAGGI, of the University of Pisa, has been elected corresponding member of the mathematical society of Charkow.

AT the University of Chicago, Professor O. BOLZA has returned from a year's leave of absence and will resume his duties commencing with the autumn quarter; Professor L. E. DICKSON has been granted leave of absence, and will spend the present year in Europe.

IN connection with the celebration of the twentieth anniversary of the founding of Clark University the honorary degree of doctor of laws was conferred upon Professor W. F. OSGOOD, of Harvard University, Professor J. PIERPONT, of Yale University, and Professor E. B. VAN VLECK, of the University of Wisconsin; the degree of doctor of mathematics was conferred upon Professor E. H. MOORE, of the University of Chicago.

AT Northwestern University Professor D. R. CURTISS has been promoted to a full professorship of mathematics. Mr. R. E. WILSON and Dr. J. C. MOREHEAD have been promoted to assistant professorships of mathematics.

PROFESSOR G. W. HARTWELL, of the University of Kansas, has accepted the professorship of mathematics at Hamline University, St. Paul, Minnesota.

PROFESSOR E. L. HANCOCK, of Purdue University, has been appointed professor of applied mechanics at the Worcester Polytechnic Institute.

PROFESSOR CHARLES HASEMAN, of the University of Indiana, has accepted an associate professorship of mathematics at the University of Nevada, Reno, Nevada.

PROFESSOR C. GUNDERSEN, of the Michigan Agricultural College, has accepted the professorship of mathematics at the Agricultural College of Oklahoma, Stillwater Oklahoma.

PROFESSOR O. T. GECKLER, of the Georgia School of Technology, has been appointed professor of mathematics at Whitman College, Walla Walla, Washington.

PROFESSOR A. B. FRIZELL, of Midland College, and Miss W. BAUER have been appointed instructors in mathematics at the University of Kansas.

DR. H. F. BURGESS has been appointed instructor in mathematics at the University of Wisconsin.

MR. E. F. A. CAREY has been appointed instructor in mathematics at the University of Montana.

MR. J. C. RAYWORTH has been appointed instructor in mathematics at Washington University, St. Louis.

DR. G. F. GUNDELFINGER has been appointed instructor in mathematics at the Sheffield scientific school of Yale University.

DR. L. S. DEDERICK has been appointed instructor in mathematics at Princeton University.

AT the University of Illinois, Dr. R. L. BÖRGER has been promoted to the position of associate in mathematics. Mr. G. H. SCOTT has been appointed assistant in mathematics.

MR. J. H. KINDLE and Mr. J. R. TRIMBLE have been appointed instructors in mathematics at the University of Cincinnati.

MR. G. H. PALMER has been appointed instructor in mathematics and graphics at Rutgers College.

MR. F. C. EATON has been appointed instructor in mathematics at the University of Iowa.

MR. R. K. WILLIAMS has been appointed instructor in mathematics at Indiana University.

AT the University of Maine Mr. S. D. CHAMBERS and Mr. T. L. HAMLIN have been appointed instructors in mathematics.

AT the Western Reserve University, Cleveland, Ohio, Professor C. J. SMITH has been granted leave of absence for the current year. Mr. A. H. FORD has been appointed instructor in mathematics.

PROFESSOR C. REUSCHLE, of the technical school at Stuttgart, died August 17, at the age of 62 years.

PROFESSOR IRVING STRINGHAM, dean of the University of California since 1886, professor of mathematics since 1882, and acting president during the current year, died on October 5, 1909, at the age of 52 years. He was graduated at Harvard in 1877, and took the degree of doctor of philosophy at Johns Hopkins in 1880. Professor Stringham became a member of the American Mathematical Society in 1891, and was a member of the Council in 1902–1905 and Vice-President of the Society in 1906.

CATALOGUES of second-hand books: Bowes and Bowes, 1 Trinity St., Cambridge, England, catalogue No. 326, containing 2576 titles in pure and applied mathematics. — W. Heffer and Sons, Cambridge, catalogue No. 53, containing titles 287–1111 on mathematics and physics. — Meier and Ehrat, Untere Bahnhofstrasse 94, Zurich, Switzerland, catalogue No. 300, about 650 titles in mathematics. — Martin Breslauer, Unter den Linden 16, Berlin W. 64, Germany, catalogue No. 1, alte Rechenbücher, etc., 29 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

DAIBER (E.). Die Formänderung rechter Winkel. (Diss.) Stuttgart, 1909. 8vo. 78 pp.

DIENES (P.). Essai sur les singularités des fonctions analytiques. (Thèse.) Paris, Gauthier-Villars, 1909. 4to. 91 pp.

FINE (H. B.). and THOMPSON (H. D.). Coördinate geometry. New York, Macmillan, 1909. 8vo. 8 + 300 pp. Cloth. $1.60

FRANKENBACH (F. W.). Lineare Erzeugung der Kegelschnitte und auf ihr beruhende Ableitung der Kegelschnittsgleichungen. Ein Beitrag zur Lehre von den Kurven 2ter Ordnung zum Gebrauch an Realanstalten. Liegnitz, Krumbhaar, 1909. 8vo. 49 pp. M. 1.00

GRASSMANN (H.). Projektive Geometrie der Ebene, unter Benützung der Punktrechnung dargestellt. Erster Band: Binäres. Leipzig, Teubner, 1909. 8vo. 12 + 360 pp. Cloth. M. 13.00

HÖEGH (E.). Elementarer Beweis des Fermat'schen Satzes. Rostock, Boldt, 1909. 8vo. 8pp. M. 1.50

MURER (V.). Introduzione alla teoria dei numeri, con numerosi esercizi e con notizie storiche. Livorno, Giusti, 1909. 16mo. 6 + 123 pp. L. 1.00

PERRON (O.). Ueber eine Verallgemeinerung des Stolzschen Irrationalitätssatzes. München, 1909. 8vo. 18 pp.

PRINZHORN (H.). Der Fermatsche Satz und sein Beweis. Magdeburg, Peters, 1909. 8vo. 11 pp. M. 1.00

SCHANZ (J.). Der Aufbau des komplexen Zahlengebiets auf der natürlichen Zahlenreihe. (Progr.) Berlin, 1909. 8vo. 31 pp.

SPEISER (A.). Die Theorie der binären quadratischen Formen mit Koefficienten und Unbestimmten in einem beliebigen Zahlenkörper. (Diss.) Göttingen, 1909.

THOMPSON (H. D.). See FINE (H. B.).

WETTERNIK (J.). Divergente Reihen und deren Anwendung auf lineare Differenzialgleichungen. (Progr.) Wien, 1908. 8vo. 16 pp.

ZIEMKE (E.) Ueber partielle Differentialgleichungen erster Ordnung mit Integralvereinen, die als Punktmannigfaltigkeiten zweifach ausgedehnt sind. (Diss., Greifswald.) Leipzig, 1909.

II. ELEMENTARY MATHEMATICS.

BALSER (L.). See NELL (A. M.).

BAUDOIN (P.). Cahiers d'exécution de dessins géométriques d'après le cours abrégé de géométrie de M. Carlo Bourlet. 1er cahier. Classes de 6e, 5e et 4e B. Paris, Hachette, 1909. 8vo. 48 pp. Fr. 1.50

BAUR (L.). Lehr- und Uebungsbuch der allgemeinen Arithmetik und Algebra. Resultate. 2te Auflage. Stuttgart, Bonz, 1909. 8vo. 60 pp. Boards. M. 1.60

BOYMAN (J. R.). Lehrbuch der Mathematik für Gymnasien, Realschulen und andere höhere Lehranstalten. Teil II: Geometrie der Ebene, ebene Trigonometrie und Geometrie des Raumes. Düsseldorf, Schwann, 1909. 8vo. 12 + 520 pp. Cloth. M. 4.80

BÜTZBERGER (F.). Lehrbuch der ebenen Trigonometrie mit vielen Aufgaben und Anwendungen für Gymnasien, Seminarien und technische Mittelschulen, sowie zum Selbstunterricht. 4te verbesserte und vermehrte Auflage. Zürich, Füssli, 1909. 8vo. 12 + 84 pp. Cloth. M. 2.00

CAPPILLERI (K.). Die Einführung der Infinitesimalrechnung in der Realschule. (Progr.) Wien, 1908. 8vo. 33 pp.

CRATHORNE (A. R.). See RIETZ (H. L.).

EDERT (R.) und KRÖGER (M.). Geometrie für Mittelschulen und verwandte Anstalten. Heft 1. Vorkursus und Planimetrie. 2te unveränderte Ausgabe. Hannover, Meyer, 1909. 8vo. 7 + 91 pp. Boards. M. 1.20

FERVAL (H.). Cours de géométrie cotée à l'usage des candidats à l'Ecole spéciale militaire de Saint-Cyr. 3e édition. Paris, Belin frères, 1909. 8vo. 248 pp. Fr. 4.00

GODFREY (C.) and SIDDONS (A. W.). Solutions of the exercises in modern geometry. Cambridge, University Press, 1909. 8vo. 124 pp. 4 s.

HARTL (H.). Lehrbuch der Planimetrie. Für den Unterrichtsgebrauch und für das Selbststudium verfasst. 2te verbesserte Auflage. Wien, Deuticke, 1909. 8vo. 5 + 141 pp. Cloth. M. 2.40

HEILERMANN und DIEKERMANN's Lehr- und Uebungsbuch für den Unterricht in der Algebra an den höheren Schulen. Neu bearbeitet von K. Knops. Essen, Baedeker, 1909. 8vo. 8 + 265 pp. Cloth. M. 2.80

KARST (L.). Lineare Funktionen und Gleichungen. (Progr.) Lichtenberg bei Berlin, 1909. 4to. 44 pp.

KOPPE und DIEKMANN's Geometrie zum Gebrauche an höheren Unterrichtsanstalten. Teil III. Die Stereometrie, der Koordinatenbegriff, die Kegelschnitte. 3te Auflage, bearbeitet von K. Knops. Essen, Baedeker, 1909. 8vo. 4 + 168 pp. Cloth. M. 2.80

KRÖGER (M.). See EDERT (R.).

MAHLERT (A.). See MÜLLER (H.).

MILNE (W. J.). Key to standard algebra. New York, American Book Co., 1909. 12mo. 432 pp. Cloth. $1.00

MÜLLER (H.) und MAHLERT (A.). Mathematisches Lehr- und Uebungsbuch für das Lyzeum. Fortsetzung des mathematischen Lehr- und Uebungsbuches für höhere Mädchenschulen. 2ter Teil. Leipzig, Teubner, 1909. 8vo. 8 + 339 pp. Cloth. M. 4.40

NELL (A. M.). Fünfstellige Logarithmen der Zahlen und der trigonometrischen Functionen. 13te Auflage von L. Balser. Giessen, Roth, 1909. 8vo. 6 + 84 pp. Cloth. M. 2.00

OTTO (F.) und SIEMON (P.). Lehr- und Uebungsbuch der Geometrie für zehnklassige höhere Mädchenschulen. Leipzig, Hirt, 1909. 8vo. 144 pp. Cloth. M. 1.50

PIERPONT (A. E.). The elements of geometry in theory and practice. Based on the report of the committee appointed by the mathematical association. London, Longmans, 1909. 8vo. 3 s.

RIETZ (H. L.) and CRATHORNE (A. R.). College Algebra. New York, Holt, 1909. 12mo. $1.40

SIDDONS (A. W.). See GODFREY (C. J.

SIEMON (P.). See OTTO (F.).

VALLÍN Y BUSTILLO (A. F.). Geometría para los niños. Madrid, Hernando, 1909. 8vo. 122 pp. P. 1.25

III. APPLIED MATHEMATICS.

BLOMFIELD (C. H.). See JONES (A. C.).

GASCUE (F.). Curso elemental de mecánica y construcción. 3a edición. 2 vols. Sama de Langreo, 1909. 8vo. 15 + 333 pp. P. 11.00

GRIMSEHL (E.). Lehrbuch der Physik. Zum Gebrauch beim Unterricht, bei akademischen Vorlesungen und zum Selbststudium. Mit 1091 Figuren im Text, 2 farbigen Tafeln und einem Anhange, enthaltend Tabellen physikaljscher Konstanten und Zahlentabellen. Leipzig, Teubner, 1909. M. 16.00

JONES (A. C.) and BLOMFIELD (C. H.). Elementary mechanics of solids and fluids. London, Arnold, 1909. 8vo. 388 pp. 4s. 6d.

LEBLANC (H.). Les mécanismes. Traité élémentaire de cinématique appliquée. 2e édition. Paris, Garnier frères, 1909. 18mo. 672 pp.

MANVILLE (O.). Les découvertes modernes en physique. Première partie. Electricité et matière. Deuxième partie, les ions et les électrons dans la théorie des phénomènes physiques; la matière et l'éther. Deuxième édition revue et augmentée. Avec 65 figures dans le texte. Paris, Hermann, 1909. 8vo. 463 pp. Fr. 8.00

MARTINI (Z. A.). Teoria matematica del conto corrente e sue applicazioni. Livorno, Giusti, 1909. 16mo. 59 pp. L. 0.50

MARVÁ Y MAYER (J.). Mecánica aplicada á las construcciones. 2 vols. 4a edición. Madrid, Palacio, 1909. 8vo. 2042 pp. P. 40.00

MÜLLER (J.). Nautik. Mit 58 Figuren im Text und auf einer Tafel. Leipzig, Teubner, 1909. M. 1 25

PILKINGTON (W.). Coördinate geometry applied to land-surveying. London, Spon, 1909. 12mo. 44 pp. 1s. 6d.

TESAR (L.). Die Mechanik. Eine Einführung mit einem metaphysischen Nachwort. Mit 111 Figuren. Leipzig, Teubner, 1909.

VERGNE (H.). Contribution à la théorie des ondes liquides. (Thèse.) Paris, Gauthier-Villars, 1909. 4to. 85 pp.

WILSON (V. T.). Descriptive geometry: a treatise from a mathematical standpoint, together with a collection of exercises and practical applications. New York, Wiley, 1909. 8vo. Cloth. $1.50